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On the Difference between TFR and Parity Progression Measure of Fertility

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Abstract

This brief note analytically examines so called "parity distribution effect," namely the difference between the TFR and PAP (Period Average Parity), the latter a life table measure of fertility. It is shown that the difference never appears in a Poisson process, which stems from the assumption that the intensity of birth is independent from parity. Analyses assuming exponential or geometric distribution of parity show that the PAP is more robust and less likely to exaggerate fertility decline than the TFR. The smaller difference in Japan than in the Republic of Korea could be attributed to slower delay in childbearing and a higher proportion of childless women in Japan, in addition to possible measurement error due to the lack of fertility data in the Japanese census.

Introduction

This brief note compares the TFR (Total Fertility Rate) and PAP (Period Average Parity), the latter a life table measure of parity progression process. While the TFR is based on so called "incidence rates," the PAP refers to hazard or "intensity of birth" which is theoretically more favorable. The difference between the TFR and PAP can be regarded as "parity distribution effect" without considering tempo distortion (Ortega and Kohler, 2002, pp. 17-18). It is considered that the PAP is based on a rate that is theoretically more favorable but also is more robust to tempo effects (Bongaarts and Feeney, 1998, p. 274). In analytically examining the characteristics of PAP in comparison with TFR, it is assumed that the number of births has specific distributions such as Poisson, exponential, and geometric distributions. In a stable state wherein all cohorts have the exactly same fertility schedule, no difference between the TFR and PAP appears. To introduce perturbation, two scenarios are considered. One is a one-time age-neutral quantum decline, and the other is a one-time age-shift without any change in quantum or shape.

1. Two Kinds of Fertility Rates

The denominator of the ordinary age-specific fertility rate is entire whole female population of a specific age group. This rate is referred to as the "incidence rate" (Ortega and Kohler, 2002, pp. 4-5). Even when the birth order is considered, no structuring of the female population other than age is introduced. Since the denominator is common for birth order-specific fertility rates by age, the rates are additive. The TFR can be defined as the age total of the incidence rates without considering birth order as well as the sum of the total order specific rates. If f(x,i) is the incidence rate of mother's age x and birth order i, and [α , β) is the reproductive period,

$$TFR = \int_{\alpha}^{\beta} \sum_i f(x,i)\,dx = \sum_i \int_{\alpha}^{\beta} f(x,i)\,dx.$$

To obtain the theoretically favorable hazard or "intensity of birth", considering the population at risk, the denominator should be the age-parity specific female population. Since a woman's first childbirth occurs at parity 0, the number of births of birth order i+1 is divided by the female population of parity i. If p(x,i) is the parity distribution of women of age x,

$$\sum_i p(x,i) = 1.$$

The relation between the incidence rate and the intensity of birth, m(x,i), is,

$$f(x,i) = m(x,i)\,p(x,i).$$

It is obvious that all women at the beginning of their reproductive period are classified as parity 0. Namely, $p(\alpha,0)=1$ and $p(\alpha,i)=0$ for $i>0$. A multi-state life table can be produced, based on the following system.

$$\frac{d}{dx}p(x,0) = -m(x,0)\,p(x,0),$$

$$\frac{d}{dx}p(x,i) = m(x,i-1)\,p(x,i-1) - m(x,i)\,p(x,i), \quad i > 0.$$

In practice, adequate linear or exponential interpolation should be applied. The PAP is calculated from the eventual parity distribution at the end of the reproductive period.

$$PAP = \sum_i i\ p(\beta, i).$$

Rallu and Toulemon (1993) referred to this measure as PATFR (Parity and Age Total Fertility Rate). The TFRPPR (TFR based on Parity Progression Ratio) by Feeney (1986) is also a closely related measure.

2. Introducing Perturbation

In a stable state in which all cohorts have the exactly same fertility schedule, there is no difference between TFR and PAP. To examine the difference, two scenarios of perturbation are hereby introduced. One is a universal decline in intensity of birth. Assume that while older cohorts have the fertility schedule $m(x,i)$, younger cohorts follows the schedule of $c\ m(x,i)$, where $0<c<1$. This expresses a decline in the quantum of fertility without a tempo change. If X is the age of the transitional cohort, the age pattern of intensity changes as X moves from α to β as shown in Figure 1. The TFR and PAP can be seen as the function of X and are denoted TFR(X) and PAP(X), respectively. The change starts when the transitional cohort begins childbearing ($X=\alpha$) and ends when the cohort finishes reproduction ($X=\beta$).

Another scenario is an age-shift of intensity without any change in quantum or shape. Assume that while older cohorts have the fertility schedule $m(x,i)$, younger cohorts follows the schedule of $m(x-h,i)$, where $0<h$. This expresses a delay in the tempo of fertility without any change in quantum or shape. The age pattern of intensity changes with the aging of the transitional cohort, as shown in Figure 2. As in these figures, it is assumed that intensity of birth has a unimodal age pattern. It is also assumed that the upper limit of reproduction does not change from β. This implies that, even in the age-shift of intensity, the eventual TFR and PAP are less than the original ones because fertility around the end of reproduction is lost.

3. Poisson Distribution

Here we assume that the intensity of birth is independent from parity. Namely, the intensity is a univariate function of age and can be written as $m(x)$. If $M(x)$ is the cumulative intensity,

$$M(x) = \int_0^x m(a)\ da.$$

Under this condition, parity has the Poisson distribution with the parameter $M(x)$ (Krishnamoothy, 1979).

$$p(x,i) = \frac{M(x)^i e^{-M(x)}}{i!}.$$

Since intensity is independent from parity, it turns out that the ordinary incidence rate without considering parity is equivalent to the intensity.

$$f(x,i) = m(x)\ p(x,i),$$

$$f(x) = \sum_i f(x,i) = m(x).$$

This implies that the TFR is equivalent to the sum of intensity for all ages. The PAP is defined as the average parity at the end of the reproductive period. Thus, as far as the eventual parity has the Poisson distribution which parameter is the TFR, there is no difference between the TFR and PAP.

Figure 1. Age-Neutral Quantum Decline

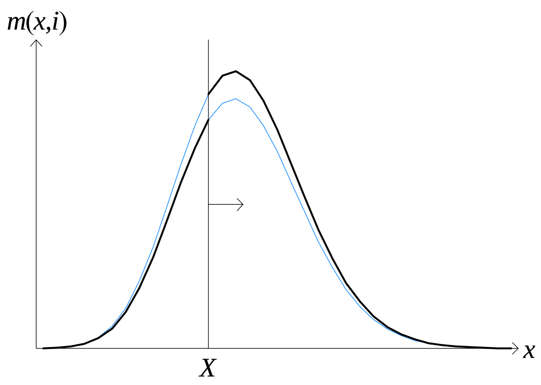
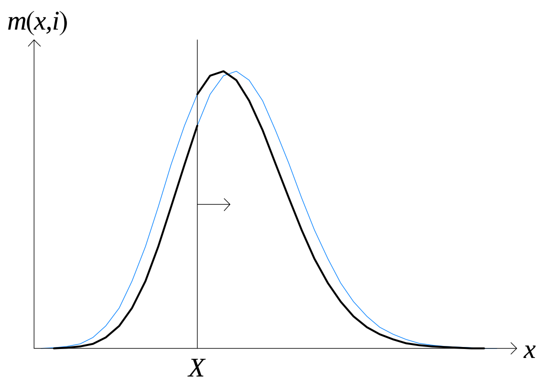


Figure 2. Delay in Intensity



Because neither scenario considered here affects the assumption of the independence of intensity with parity, the parity distribution of a hypothetical cohort always has the Poisson distribution with the parameter $M(x)$. Therefore, there is no parity distribution effect in a Poisson process.

4. Exponential Distribution

For the sake of simplicity, this section concentrates on first births. Therefore, the TFR represents the average number of first children and the PAP the proportion of women who have ever given birth. For the sake of convenience, the intensity of first birth is written as $m_0(x)$ and its cumulative as $M_0(x)$.

$$M_0(x) = \int_0^x m_0(a)\, da.$$

The proportion of women who have never given birth, $p(x,0)$, is written as $S_0(x)$.

$$S_0(x) = p(x,0) = e^{-M_0(x)}.$$

This form can be seen as an expansion of basic exponential distribution with a constant intensity, as in the case of the Poisson distribution. Since the incidence rate is the product of the intensity and parity,

$$f_0(x) = m_0(x)\, S_0(x).$$

4-1. Age-Neutral Quantum Decline

Here we assume that there was a one-time change in intensity from $m_0(x)$ to $c\, m_0(x)$, where $0<c<1$. Since the cumulative hazard is also multiplied by c, the proportion of women who have never given birth will be powered by c for cohorts younger than the transitional cohort.

$$e^{-c\, M_0(x)} = S_0(x)^c.$$

The TFR when the transitional cohort is at age X is the original TFR minus the change in the proportion of non-mothers at age X.

$$TFR(X) = 1 - S_0(\beta) - \{S_0(X)^c - S_0(X)\}.$$

For the PAP, the ratio of proportion of childless women, rather than the difference, is the matter of consequence.

$$PAP(X) = 1 - S_0(\beta) \frac{S_0(X)^c}{S_0(X)}.$$

It can be shown that the PAP declines monotonously as the transitional cohort ages.

$$\frac{d}{dX} PAP(X) = -(1-c)\, m_0(X)\, S_0(\beta) \frac{S_0(X)^c}{S_0(X)} < 0.$$

On the other hand, the TFR can have an extreme value within the reproductive period to produce a U-shaped trajectory.

$$\frac{d}{dX} TFR(X) = c\, m_0(X)\, S_0(X)^c - m_0(X) S_0(X).$$

The above equation implies that TFR(X) hits the bottom when the old and new incidence rates of first births are equivalent. If X^* is the age corresponds to the bottom of trajectory,

$$c\, e^{-c\, M_0(X^*)} = e^{-M_0(X^*)}.$$

This produces the following solution.

$$M_0(X^*) = -\frac{\ln c}{1-c}.$$

Since the right hand side is greater than one, $TFR(X)$ does not have an extreme value within the reproductive period if $M_0(\beta) \leqq 1$. In such a case, TFR(X) will decline monotonously. For the TFR to produce a U-shaped trajectory, the cumulative hazard must exceed the unity before the reproduction ends. This implies that the proportion of childless women should be smaller than exp(-1) which is approximately 36.8 per cent. This is a weak requirement and the TFR will almost always produce a U-shaped trajectory except in case of very low fertility.

It is obvious that $PAP(\alpha) = TFR(\alpha)$ and $PAP(\beta) = TFR(\beta)$. If the PAP declines monotonously and the TFR follows a U-shaped trajectory with only one extreme value, it is expected that the PAP is greater or equal to the TFR throughout the change.

4-2. Age-Shift of the Intensity

Here we assume that the intensity of birth for cohorts younger than the transitional cohort has shifted, such that $m_0(x) \rightarrow m_0(x-h)$ where $h>0$. Since the cumulative hazard also shifts by h, the proportion of childless women shifts by h.

$$e^{-M_0(x-h)} = S_0(x-h).$$

The TFR and PAP have the same form as in the case of quantum decline but $S_0(X)^c$ is replaced by $S_0(X-h)$.

$$TFR(X) = 1 - S_0(\beta) - \{S_0(X-h) - S_0(X)\}.$$

$$PAP(X) = 1 - S_0(\beta)\frac{S_0(X-h)}{S_0(X)}.$$

The following result shows that $PAP(X)$ produces a U-shaped trajectory because the first parenthesis shifts from negative to positive under the assumption of the unimodal age pattern of $m_0(x)$.

$$\frac{d}{dX}PAP(X) = \{m_0(X-h) - m_0(X)\} \cdot S_0(\beta)\frac{S_0(X-h)}{S_0(X)}.$$

It turns out that the derivative of $TFR(X)$ is simply the difference between incidence rates. Thus, the TFR also produces a U-shaped trajectory.

$$\frac{d}{dX}TFR(X) = f_0(X-h) - f_0(X).$$

It can be shown that the TFR is smaller than the PAP. In the following result, the equation on the right hand side is a quadratic equation of $S_0(X)$ whose roots are $S_0(X-h)$ and $S_0(\beta)$. Because $S_0(X)$ is between $S_0(X-h)$ and $S_0(\beta)$, the right hand side should be always non-negative. This implies that $PAP(X) \geqq TFR(X)$.

S0(X) {PAP(X) - TFR(X)} = -S0(X)2
+ {S0(β)+S0(X-h)}S0(X)
- S0(β) S0(X-h).

5. Geometric Distribution

Two important results were obtained in the analysis of first births in the former section. Firstly, the TFR can produce a U-shaped trajectory even in the case of age-neutral quantum decline. Secondly, the TFR is smaller than the PAP throughout the change. These results could be sustained in a more general setting of multiple births if some assumptions were made on parity distribution. To illustrate an example, it is assumed here that parity has the geometric distribution at all ages. Although the reality of this assumption is questionable, this will produce the simplest solution.

As in the branching process (Harris, 1989, p. 9), it is assumed that the proportion of childlessness is given exogeneously. If $R(x)$ is the parity progression ratio at age x and for $i>0$,

$$p(x,i) = \{1 - p(x,0)\}\,\{1 - R(x)\}\,R(x)^{i-1}.$$

Recall that the mean of geometric distribution is $1/\{1-R(x)\}$. To simulate low fertility in contemporary Japan, it is desirable that the conditional mean for women who ever had birth, $1/\{1-R(x)\}$, declines monotonously within the range between one and two. Here we choose the following form.

$$\frac{1}{1-R(x)} = 2 - p(x,0),$$

$$\text{or,}\quad R(x) = \frac{1-p(x,0)}{2-p(x,0)}.$$

Then, parity has the following distribution at age x.

$$p(x,i) = \left(\frac{1-p(x,0)}{2-p(x,0)}\right)^i, \quad i>0.$$

As in the previous section, it is assumed that $S_0(x) = p(x,0)$ is determined by the intensity $m_0(x)$ and its cumulative function $M_0(x)$. The general system of parity progression is,

$$\frac{d}{dx}p(x,0) = -m(x,0)p(x,0),$$

$$\frac{d}{dx}p(x,i) = m(x,i-1)p(x,i-1) - m(x,i)p(x,i),\ i>0.$$

Applying the assumption of geometric distribution yields the following solution.

$$f(x,i) = m(x,i)p(x,i) = S_0(x)m_0(x)\frac{\{1-S_0(x)\}^i\{i+2-S_0(x)\}}{\{2-S_0(x)\}^{i-1}}.$$

$$f(x) = \sum_i f(x,i) = S_0(x)\,m_0(x)\,\{3 - 2\,S_0(x)\}.$$

In a stable state,

TFR= PAP= {1 - S0(β)} {2 - S0(β)}
= S0(β) 2 - 3 S0(β) + 2.

5-1. Age-Neutral Quantum Decline

Here it is assumed, as in Section 4-1, that there has been change from $m_0(x)$ to $c\ m_0(x)$, where $0<c<1$. For cohorts at age X and below, the proportion of childlessness is $S_0(X)^c$. The new incidence rate for all parities is,

$$f_2(x) = c\ m_0(x)\ S_0(x)^c\ \{3 - 2\ S_0(x)^c\}.$$

The difference in $S_0(X)$ and its square causes the TFR change in an additional way. On the other hand, each argument of the original PAP is multiplied with the ratio of $S_0(X)$ and its square.

TFR(X) = S0(β) 2 - 3 S0(β) + 2
- {S0(X) 2 - S0(X) 2c}
+ 3{S0(X) - S0(X) c}.

$$PAP(X) = S_0(\beta)^2 \frac{S_0(X)^{2c}}{S_0(X)^2} - 3S_0(\beta)\frac{S_0(X)^c}{S_0(X)} + 2.$$

As in Section 4-1, the PAP declines monotonously.

$$\frac{d}{dX} PAP(X) = -(1-c)\ f_0(X) S_0(\beta) \cdot \frac{S_0(X)^c}{S_0(X)^2}\left\{3 - 2S_0(\beta)\frac{S_0(X)^c}{S_0(X)}\right\} < 0.$$

The TFR hits the bottom when the old and new incidence rates are equivalent.

$$\frac{d}{dX} TFR(X) = \frac{f_0(X)}{m_0(X)S_0(X)}\{f_2(X) - f(X)\}.$$

As in Section 4-1, the TFR declines monotonously if fertility is extremely low. The condition $f_2(X^*) = f(X^*)$ is equivalent with the following equation of $S_0(X^*)$.

$$\frac{S_0(X^*)^2 - c\,S_0(X^*)^{2c}}{S_0(X^*) - c\,S_0(X^*)^c} = \frac{3}{2}.$$

One asymptotic line is $S_0(X^*) = c^{1/(1-c)}$ which makes the denominator zero. Within the range $c^{1/(1-c)} < S_0(X^*) \leqq 1$, the maximum of the equation on the left hand side represents the unity when $S_0(X^*) = 1$. Thus, the condition $f_2(X^*) = f(X^*)$ is satisfied only when $S_0(X^*) < c^{1/(1-c)}$. If fertility is low enough so that $c^{1/(1-c)} < S_0(\beta)$, there is no such X^* that satisfies $f_2(X^*) = f(X^*)$.

The value of $c^{1/(1-c)}$ is 0.3487 when $c = 0.9$ and 0.3277 when $c = 0.8$. Except for a very drastic change or very low fertility, there exists X^* corresponding to the extreme value. Thus, we can expect that the TFR produces a U-shaped trajectory and is lower than the PAP throughout the change.

5-2. Age-Shift of the Intensity

If there were an age-shift of the sort wherein $m_0(x) \rightarrow m_0(x-h)$, the proportion of childlessness would also shift from $S_0(x)$ to $S_0(x-h)$. Thus, the incidence rate would also shift in the same way.

$$f_2(x) = f(x-h)$$

The TFR and PAP would have the same form as in the quantum decline but $S_0(X)_c$ would be replaced by $S_0(X-h)$.

$TFR(X) = S_0(\beta)^2 - 3\ S_0(\beta) + 2$
$- \{S_0(X)\ 2 - S_0(X-h)^2\}$
$+ 3\{S_0(X) - S_0(X-h)\}.$

$$PAP(X) = S_0(\beta)^2 \frac{S_0(X-h)^2}{S_0(X)^2} - 3S_0(\beta)\frac{S_0(X-h)}{S_0(X)} + 2.$$

The PAP will produce a U-shaped trajectory in accordance with the difference between $m_0(X)$ and $m_0(X-h)$. Assume that $S_0(\beta) < 1/2$. In the following equation, the last parenthesis is positive only if $S_0(x-h)$ is more than three times as large as $S_0(x)$. Such a drastic delay is not assumed here and the PAP does not produce a reversed U-shaped curve.

$$\frac{d}{dX} PAP(X) = S_0(\beta)\frac{S_0(X-h)}{S_0(X)^2} \cdot \{m_0(X) - m_0(X-h)\} \cdot \{2S_0(\beta)S_0(X-h) - 3S_0(X)\}.$$

As in Section 4-2, the derivative of $TFR(X)$ is simply the difference between the old and new incidence rates at age X. This implies that the TFR also shows a U-shaped trajectory.

$$\frac{d}{dX} TFR(X) = f_0(X-h) - f_0(X).$$

Here we assume that $S_0(X) / S_0(X-h) < 2$ and $S_0(\beta) < 1/2$. In the last big parenthesis of the following equation, $1 + S_0(X) / S_0(X\text{-}h)$ is less than 3 and $S_0(X) + S_0(\beta)$ is less than 3/2. Thus, the PAP is larger than the TFR except for a lengthy delay or extremely low fertility.

$$PAP(X) - TFR(X)$$

$$= \left\{1 - \frac{S_0(X-h)}{S_0(X)}\right\}\{S_0(X) - S_0(\beta)\}$$

$$\cdot \left[\left\{1 + \frac{S_0(X-h)}{S_0(X)}\right\}\{S_0(X) + S_0(\beta)\} - 3\right].$$

6. Numerical Example

The analytical results so far illustrate that the TFR tends to produce a U-shaped trajectory even in a case of age-neutral quantum decline. This implies that the TFR tends to exaggerate the fertility decline. Unfortunately, both TFR and PAP produce U-shaped curves in a case of delay in childbearing. However, the PAP is more robust and the degree of exaggeration thereof is smaller than that of the TFR.

Table 1 compares the TFR and PAP in Japan and the Republic Korea in 2000. The parity distribution in Korea was obtained from the 2000 census. Because the Japanese census lacks information on fertility, the cohort specific parity distribution in the year 2000 was estimated based on a series of incidence rates of each cohort. As expected, the PAP in Korea showed higher value than the TFR. In Japan, however, the PAP was slightly lower than the TFR.

Table 1. PAP and TFR in Japan and Korea

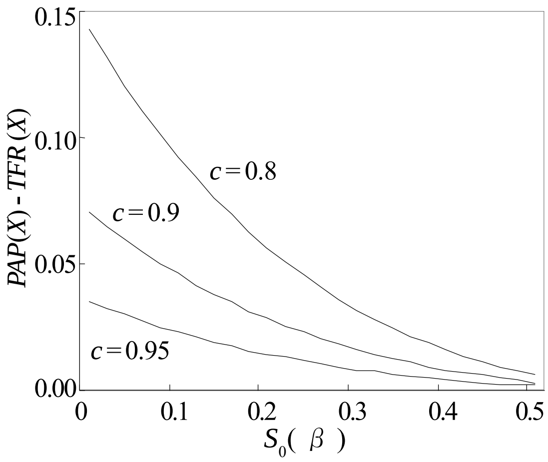
Eventual Parity Distribution	Japan (2000)	Korea (2000)
0	0.2958	0.1555
1	0.2077	0.2460
2	0.3685	0.5055
3	0.1073	0.0851
4+	0.0208	0.0078
PAP	1.35	1.54
TFR	1.36	1.47

If there were any measurement errors, the result on Japan would be more problematic because the parity distribution was estimated indirectly. However, there are some reasons to believe that the difference between the TFR and PAP should be smaller in Japan than in Korea.

One reason is Japan's slow delay in childbearing. The average age at childbearing rose from 29.0 in 1990 to 29.7 in 2000. In the same period, however, the average age rose from 27.2 to 28.9 in Korea. In fact, the tempo change in Japan is among the slowest in countries suffering from very low fertility. Many European countries with low fertility experienced a delay of one year over a period of six years or less (Suzuki, 2003, p. 4). It is apparent that the slower the delay, the smaller the parity distribution effect.

Another reason is the high proportion of eventual childlessness, $S_0(\beta)$, in Japan. As Table 1 reveals, the proportion was 29.6% in Japan in 2000. This is not the extremely low level of fertility that can cause a monotonous decline of the TFR in the case of quantum decline, a reversed U-shaped trajectory of the PAP in postponement of childbearing, or a higher value of the PAP than the TFR. However, it can be shown that the higher the proportion of childless women, the smaller the parity distribution effect. Figure 3 shows an example when $S_0(X)$ is at the midpoint of its change in the scenario of age-neutral quantum decline.

Figure 3. The Difference between TFR and PAP by $S_0(\beta)$ and c in Quantum Decline



Conclusion

This paper has compared the TFR and PAP in terms of their responses to one time quantum or tempo change. It has been shown here that the TFR tends to exaggerate fertility decline and to show lower value than the PAP. It is also shown herein that the slow change in tempo and high proportion of childlessness could explain the small difference between the TFR and PAP in Japan. The numerical example employed was for the year 2000, preceding a drastic fertility decline in Korea. It will be interesting to see if the difference in Korea in 2005 decreased, as would be expected from the results of this paper.

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